

## Power Convergent Boolean Matrices\*

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### ABSTRACT

An  $n \times n$  binary matrix  $A$  is *limit dominating* if its powers converge to a limit  $L$  and  $A \geq L$ . (The inequality holds entrywise, and arithmetic on all binary matrices is assumed to be Boolean:  $1 + 1 = 1$ .) Limit dominating matrices generalize transitive matrices ( $A \geq A^2$ ). If  $A$  is limit dominating, we show that the power limit  $L$  can be found immediately from  $A$  and that the *residual* matrix  $N = A \setminus L$  is nilpotent. Moreover,  $A^k = L + N^k$  for all positive integers  $k$ . We prove that  $\nu \geq \kappa \geq n/(n - \nu + 1)$ , where  $\kappa$  is the index of convergence of  $A$  (the first integer  $k$  such that  $A^k = L$ ), and  $\nu$  is the index of convergence of  $N$ . We characterize an extreme class of limit dominating matrices, the idempotent matrices ( $A^2 = A$ ). In particular, we show that a binary matrix  $A$  is idempotent if and only if it is limit dominating and the number of nonzero diagonal blocks in its Frobenius normal form equals its column rank (as a Boolean matrix). Finally, we give natural generalizations to matrices with entries from an arbitrary finite Boolean algebra.

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### 1. INTRODUCTION

We first consider the *binary* Boolean matrices. They are matrices whose entries are 0 or 1; the arithmetic underlying the matrix multiplication and addition is Boolean, that is, it is the usual integer arithmetic except that  $1 + 1 = 1$ . We denote the rows of an  $m \times n$  binary matrix  $A$  by  $A_i$ , the columns by  $A_j$ , and the entries by  $a_{ij}$  for  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ . If  $A$  and  $B$

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are binary matrices of the same size, then we write  $A \geq B$  (and say that  $A$  *dominates*  $B$ ) if the inequality holds entrywise. When  $A \geq B$ , we write  $A \setminus B$  for the matrix whose  $(i, j)$  entry is 1 if  $a_{ij} = 1$  and  $b_{ij} = 0$ , and is 0 otherwise.

A square binary (Boolean) matrix  $A$  is *power convergent* if its powers  $A^k$  converge. It is *limit dominating* if it is power convergent and dominates its power limit. By a direct argument, or by using Theorem 1.1 below, one can show that  $A$  is limit dominating if and only if  $A \geq A^k$  for all sufficiently large  $k$ . Consequently, limit dominating binary matrices can be regarded as generalizations of the *transitive* matrices, that is, of those matrices  $A$  such that  $A \geq A^2$  (and so  $A \geq A^k$  for all  $k \geq 1$ ). A transitive matrix  $A$  is associated with a transitive relation  $R$  (hence the name):  $a_{ij} = 1$  if and only if  $iRj$ . A special subset of the transitive matrices are the *idempotent* matrices, those square binary matrices  $A$  such that  $A = A^2$ .

A square binary matrix  $A$  is *block upper-triangular* if it equals a block partitioned matrix with square blocks on the main diagonal and zero blocks below. The matrix  $A$  is *reducible* if there is a permutation matrix  $P$  such that  $PAP^t$  is block upper-triangular with two or more diagonal blocks; otherwise  $A$  is *irreducible*. In particular, both of the  $1 \times 1$  binary matrices are irreducible. The matrix  $A$  is *primitive* if some power of  $A$  is the all-1's matrix  $J$ .

If  $A$  is a square binary matrix, there is always a permutation matrix  $P$  such that  $PAP^t$  is block upper-triangular with irreducible diagonal blocks. This *Frobenius normal form* is perhaps most easily seen by considering the associated digraph  $D$  with adjacency matrix  $A$ : the row/column indices of the diagonal blocks correspond to the vertices of the strong components of  $D$ , and the blocks are ordered inductively by choosing a component in  $D$  with no outward arcs. (For a discussion of matrices and digraphs, see [1, Chapter 3].) The following theorem on powers of matrices can also be proved on the associated digraph, using for example Theorem 2 of [7].

**THEOREM 1.1.** *The powers of a binary matrix  $A$  converge if and only if each of the diagonal blocks of its Frobenius normal form is either primitive or a  $1 \times 1$  zero matrix.*

The only irreducible limit dominating matrices are the  $1 \times 1$  zero matrix and  $J$ . On the other hand, if  $A$  is limit dominating and reducible, then each of the irreducible diagonal blocks in its Frobenius normal form is limit dominating, and so is either a  $1 \times 1$  zero block or a  $J$  block. Indeed, every block in the normal form of  $A$  is either an all-zero block  $O$  or a  $J$  block. To see this, consider a nonzero off-diagonal block  $B_{ij}$  with  $i < j$  (if  $i > j$  then  $B_{ij} = O$ ), as well as the corresponding diagonal blocks  $B_{ii}$  and  $B_{jj}$ . If both  $B_{ii}$  and  $B_{jj}$  are  $1 \times 1$  blocks, then the claim certainly holds. So suppose that

at least one of them is a  $J$  block. Let

$$C = \begin{bmatrix} B_{ii} & B_{ij} \\ O & B_{jj} \end{bmatrix}.$$

It is easy to check that the superdiagonal block of  $C^k$  is a  $J$  block when  $k \geq 3$ . But the superdiagonal block of  $C^r$  is dominated by the corresponding block of  $A^r$  for all  $r$ . Thus the corresponding block of  $L$ , and hence of  $A \geq L$ , is a  $J$  block. Therefore the claim still holds. We now have the following theorem.

**THEOREM 1.2.** *If a binary matrix  $A$  is limit dominating, then each of the blocks of its Frobenius normal form is either a  $O$  block or a  $J$  block, and any diagonal  $O$  block is  $1 \times 1$ .*

The converse of Theorem 1.2 is false for  $n \geq 3$ . For example, take  $A = I_n + S_n$ , where  $S_n$  is the  $n \times n$  matrix that has all  $n - 1$  of its entries on the first superdiagonal equal to 1 and all other entries equal to zero. Then  $A$  is in Frobenius normal form, and its diagonal blocks are all  $1 \times 1$ . However, its power limit is  $I_n + U_n$ , where  $U_n$  is the  $n \times n$  matrix with 1's in all  $n(n - 1)/2$  positions strictly above the main diagonal and 0's on or below the main diagonal.

In Section 2, we show that if an  $n \times n$  matrix  $A$  is limit dominating, then its power limit  $L$  must equal  $\sum_{j \in \mathcal{D}} A_{\cdot j} A_{j \cdot}$ , where  $\mathcal{D} = \{j : a_{jj} = 1\}$ . Moreover, the residual matrix  $N = A \setminus L$  is nilpotent and  $A^k = L + N^k$  for all positive integers  $k$  (Theorem 2.1). Using the Frobenius normal form, we obtain upper and lower bounds on the index of convergence  $\kappa(A)$  (the first  $k$  such that  $A^k = L$ ) in terms of the size  $n$  of  $A$  and the index of nilpotence  $\nu(A)$  of its nilpotent residual (the first  $k$  such that  $N^k = O$ ). Examples 2.1 and 2.2 show that the bounds on  $\kappa(A)$  are attainable.

In Section 3, we characterize those limit dominating matrices which are idempotent. The results are motivated by earlier work of Rosenblatt [8], Schein [9], and Chaudhuri and Mukherjea [3]; the last two of these papers examine idempotence only in the context of binary relations. Theorem 3.1 states that  $A$  is idempotent if and only if it is limit dominating and the number  $\tau(A)$  of diagonal  $J$  blocks in its Frobenius normal form equals any of the following three parameters: the column rank  $c(A)$ , the Boolean rank  $b(A)$ , or the maximum number  $\iota(A)$  of isolated 1's in  $A$ . The parameters  $c(A)$ ,  $b(A)$ ,  $\iota(A)$  are defined in Section 3.

In Section 4, we consider limit dominating matrices  $A$  with entries from a finite Boolean algebra  $\mathcal{B}$  with atoms  $\alpha_i$ ,  $i = 1, \dots, m$ . Each matrix  $M$  over  $\mathcal{B}$  can be expressed as a sum  $M = \sum_{i=1}^m \alpha_i M_i$  where each constituent matrix

$M_i$  is binary. We observe that constituent matrix decompositions preserve all the matrix operations and relations that we require and use this device to extend all of our results to the general Boolean case.

Throughout the paper, we denote the  $n \times n$  identity matrix by  $I_n$ , the  $m \times n$  zero matrix by  $O_{m \times n}$ , and the  $m \times n$  all-1's matrix by  $J_{m \times n}$ , suppressing the subscripts whenever the context makes the size of the matrix clear.

## 2. POWERS OF LIMIT DOMINATING BINARY BOOLEAN MATRICES

Throughout this section and the next, *matrices* are binary Boolean matrices. A square matrix  $A$  is *limit dominating* if its powers converge to a limit  $L$  and  $A \geq L$ . Two extreme classes of limit dominating matrices are the *idempotent* matrices ( $M^k = M$  for all  $k \geq 1$ ) and the *nilpotent* matrices ( $M^k = O$  for some  $k \geq 1$ ). Our first theorem implies that every limit dominating matrix  $A$  can be expressed as the sum of two such matrices and gives an explicit formula for the power limit  $L$ .

If  $M$  is an idempotent matrix that is dominated by a power convergent matrix  $A$ , then  $M = M^k \leq A^k$  for any  $k \geq 1$ , so that  $M$  is dominated by the limit  $L$  of the powers of  $A$ . But  $L$  itself is idempotent. Therefore:

*If  $A$  is a limit dominating matrix, then its power limit  $L$  is the maximum idempotent matrix dominated by  $A$ .*

Consequently, if we are to express  $A$  as the sum of an idempotent matrix and a nilpotent one,  $L$  is a natural choice for an idempotent summand.

**THEOREM 2.1.** *Suppose that  $A$  is an  $n \times n$  limit dominating matrix and that  $L$  is the limit of the powers of  $A$ . Let  $N$  be the residual matrix  $A \setminus L$ . Then*

$$A^k = L + N^k \quad \text{for all } k \geq 1,$$

*and  $N$  is nilpotent. Moreover,*

$$L = \sum_{j=1}^n a_{jj} A_{\cdot j} A_j = \sum_{j \in \mathcal{D}} A_{\cdot j} A_j, \quad \text{where } \mathcal{D} = \{j : a_{jj} = 1\}.$$

*(The latter sum is taken to be  $O$  if  $\mathcal{D} = \emptyset$ .)*

*Proof.* Since  $N \leq A$ , we have  $LN \leq LA = L$  and  $NL \leq AL = L$ . It follows that  $A^k = (L + N)^k = L + N^k$  for all  $k \geq 1$ .

If  $PAP^t$  is in Frobenius normal form, then by Theorem 1.2 it agrees with  $PLP^t$  on the diagonal  $J$  blocks. Thus  $PNP^t$  is strictly upper-triangular, and so  $N$  is nilpotent.

Let  $S = \sum_{j \in \mathcal{D}} A_{.j} A_j$ . To show that  $S = L$ , we may assume that  $\mathcal{D} \neq \emptyset$ ; otherwise  $S = L = O$ . Suppose that  $l_{ij} = 1$ . Since  $L^n = L$ , we have  $l_{ip_1} l_{p_1 p_2} \cdots l_{p_{n-1} j} = 1$  for  $n+1$  subscripts  $i, p_1, p_2, \dots, p_{n-1}, j$ . At least two of these subscripts are equal. Since  $L$  is idempotent,  $l_{pq} = 1$  whenever  $l_{pr} = 1$  and  $l_{rq} = 1$ . Thus  $l_{ip} l_{pp} l_{pj} = 1$  for some  $p$ . But  $A \geq L$ , so  $a_{ip} a_{pp} a_{pj} = 1$ . Thus  $s_{ij} = 1$ . Therefore,  $S \geq L$ . On the other hand,  $S \leq \sum_{j=1}^n A_{.j} A_j = A^2$ , and  $S^2 \geq \sum_{j \in \mathcal{D}} (A_{.j} A_j)^2 = S$ . Consequently,  $S \leq A^{2k}$  for all  $k$ , and so  $S \leq L$ . ■

It is natural to ask at what point the powers of a limit dominating matrix  $A$  converge. Given a binary matrix  $M$  whose powers converge to a limit  $L$ , the *index of convergence*  $\kappa(M)$  of  $M$  is the smallest  $k$  such that  $M^k = L$ ; equivalently,  $\kappa(M)$  is the smallest  $k$  such that  $M^k = M^{k+1}$ . (If  $M$  is a nilpotent matrix, its index of convergence is also called its *index of nilpotence*.) From Theorem 2.1, we see that if  $A$  is a limit dominating matrix with power limit  $L$  and *residual* matrix  $N = A \setminus L$ , then  $\kappa(A)$  is the smallest  $k$  such that  $N^k \leq L$ . Thus, if  $A$  is limit dominating and we let  $\nu(A) = \min\{k : N^k = O\}$  (the index of nilpotence of the residual of  $A$ ), then  $\kappa(A) \leq \nu(A)$ .

To establish a tight lower bound on  $\kappa(A)$ , we require a lemma. Let  $S_n$  be the  $n \times n$  matrix that has all  $n-1$  of its entries on the first superdiagonal equal to 1 and all other entries equal to 0.

**LEMMA 2.1.** *If  $A$  is a limit dominating matrix and  $\nu(A) \geq 2$ , then there is a permutation matrix  $P$  such that the principal submatrix of  $PAP^t$  determined by the last  $\nu = \nu(A)$  rows and columns is a strictly upper-triangular matrix that dominates  $S_\nu$ .*

*Proof.* As  $N^{\nu-1} \neq O$ , there are indices  $i_1, \dots, i_\nu$  such that  $\prod_{j=1}^{\nu-1} n_{i_j i_{j+1}} = 1$ . If  $i_p = i_q$  for some  $p < q$ , then the  $(i_p, i_q)$  entry of  $N^{q-p}$  would be on the diagonal and equal to  $\prod_{j=p}^{q-1} n_{i_j i_{j+1}} = 1$  and so  $N$  could not be nilpotent. Thus the indices are distinct. Suppose that  $a_{i_p i_p} = 1$  for some  $p \leq q$ . Then  $a_{i_q i_q} = 1$  if  $p = q$ , while  $a_{i_q i_p} \prod_{j=p}^{q-1} n_{i_j i_{j+1}} = 1$  if  $p < q$ . Thus the  $(i_q, i_q)$  diagonal entry of each power of  $A^{q-p+1}$  is 1. Then  $l_{i_q i_q} = 1$  and so, by the formula for  $L$  in Theorem 2.1, row  $i_q$  and column  $i_q$  of  $N$  are both zero. But  $n_{i_q i_{q+1}}$  or  $n_{i_{q-1} i_q}$  is 1, a contradiction. Thus,  $a_{i_q i_p} = 0$  if  $p \leq q$ . ■

**THEOREM 2.2.** *If  $A$  is an  $n \times n$  limit dominating matrix, then*

$$\nu(A) \geq \kappa(A) \geq \left\lceil \frac{n}{n - \nu(A) + 1} \right\rceil.$$

*Proof.* The upper bound on  $\kappa(A)$  has been verified. If  $\nu(A) = 1$ , then  $N = O$ , so  $A = L$  is idempotent and  $\kappa(A) = 1$ . Thus the lower bound holds in this case. If  $A$  has no diagonal 1's then  $L = O$  (by Theorem 2.1), so  $A = N$  is nilpotent and  $\kappa(A) = \nu(A)$ . It is easy to check that the lower bound holds in this case too. Suppose now that  $\nu(A) \geq 2$  and that  $A$  has  $t$  diagonal 1's for some  $t \geq 1$ . We may assume that the last  $\nu$  rows and columns of  $A$  satisfy the condition in Lemma 2.1. Note that

$$N \geq T = O_{n-\nu} \oplus S_\nu, \quad \text{and so} \quad N^\kappa \geq T^\kappa = O_{n-\nu} \oplus S_\nu^\kappa.$$

The matrix  $T^\kappa$  is all zero except for the  $\kappa$ th superdiagonal, which has  $\nu - \kappa$  1's in positions  $(i, i + \kappa)$  for  $n - \nu + 1 \leq i \leq n - \kappa$ . Recall that  $\kappa$  is the smallest  $k$  such that  $N^k \leq L$ . Since  $N^\kappa \leq L$ , each of these  $\nu - \kappa$  1's in  $T^\kappa$  must be contained in a rank-1 matrix of the form  $A_{ij}A_{j\cdot}$ , where  $a_{jj} = 1$ . We claim that at most  $\kappa - 1$  of the  $\nu - \kappa$  1's in  $T^\kappa$  can be contained in a single product  $A_{ij}A_{j\cdot}$ . To see this, note that if  $M$  is a rank-1 matrix and  $m_{ij} = 1 = m_{pq}$ , then  $m_{iq}$  and  $m_{pj}$  must also be 1. Thus, if some  $A_{ij}A_{j\cdot}$  contains  $\kappa$  or more of the 1's in  $T^\kappa$ , then it has 1's in positions  $(i_1, i_1 + \kappa)$  and  $(i_2, i_2 + \kappa)$  where  $i_2 \geq i_1 + \kappa - 1$ . Consequently such an  $A_{ij}A_{j\cdot}$  must also have a 1 in the  $(i_2, i_1 + \kappa)$  position, and hence so must  $A$ . If  $i_2 \geq i_1 + \kappa$ , this contradicts the fact that the last  $\nu$  rows and columns of  $A$  form a strictly upper-triangular matrix. If  $i_2 = i_1 + \kappa - 1$ , then the 1 in position  $(i_2, i_1 + \kappa) = (i_2, i_2 + 1)$  is a superdiagonal entry. Thus it is in  $T$ , and so it is in  $N$ , since  $N \geq T$ . It is also in  $L$ , since it is in an  $A_{ij}A_{j\cdot}$ . This is a contradiction, since  $N = A \setminus L$ . Thus we see that each rank-1 summand  $A_{ij}A_{j\cdot}$  of  $L$  contains at most  $\kappa - 1$  of the 1's in  $T^\kappa$ , so that there must be at least  $(\nu - \kappa)/(\kappa - 1)$  such summands. Hence  $n \geq \nu + t \geq \nu + (\nu - \kappa)/(\kappa - 1)$ . This can be rearranged to give the lower bound in the statement of the theorem. ■

The following example shows that for any  $\kappa$  between the upper and lower bounds of Theorem 2.2, there is an  $n \times n$  transitive (and so, limit dominating) matrix  $A$  such that  $\nu(A) = \nu$  and  $\kappa(A) = \kappa$ . In this example (and the next),  $U_n$  denotes the  $n \times n$  matrix with 1's in all  $n(n-1)/2$  positions strictly above the main diagonal and 0's on or below the main diagonal.

**EXAMPLE 2.1.** Suppose that  $n \geq \nu \geq 2$ . (If  $\nu = 1$ , any  $n \times n$  idempotent matrix provides an example.) Fix  $\kappa$  so that  $\nu \geq \kappa \geq \lceil n/(n - \nu + 1) \rceil$ . Note that the lower bound on  $\kappa$  implies that  $\kappa \geq 2$ , and that  $n \geq \nu + d$ , where  $d = \lceil (\nu - \kappa)/(\kappa - 1) \rceil$ . We wish to find an  $n \times n$  transitive matrix  $A$  such that  $\kappa(A) = \kappa$  and  $\nu(A) = \nu$ . Since  $\kappa(A \oplus I) = \kappa(A)$  and  $\nu(A \oplus$

$I) = \nu(A)$ , it is sufficient to consider the case where  $n = \nu + d$ . Let  $r$  be such that  $\nu - \kappa = (d - 1)(\kappa - 1) + r$ . Then  $1 \leq r \leq \kappa - 1$ . Let  $A$  be the matrix of size  $n = \nu + d$  defined below:

$$A = \begin{bmatrix} I_d + U_d & B_{d \times \nu} \\ C_{d \times \nu}^t & U_\nu \end{bmatrix},$$

where

$$B_{d \times \nu} = \left[ O_{d \times \kappa} \mid X^{(1)} \mid X^{(2)} \mid \cdots \mid X^{(d-1)} \mid J_{d \times r} \right],$$

$$C_{d \times \nu} = \left[ J_{d \times (\kappa-1)} \mid Y^{(1)} \mid Y^{(2)} \mid \cdots \mid Y^{(d-1)} \mid O_{d \times \kappa} \right],$$

$$X^{(i)} = \begin{bmatrix} J_{i \times (\kappa-1)} \\ O_{(d-i) \times (\kappa-1)} \end{bmatrix}, \quad 1 \leq i \leq d-1,$$

$$Y^{(i)} = \begin{bmatrix} O_{i \times (\kappa-1)} \\ J_{(d-1) \times (\kappa-1)} \end{bmatrix}, \quad 1 \leq i \leq d-2, \quad \text{and} \quad Y^{(d-1)} = \begin{bmatrix} O_{(d-1) \times r} \\ J_{1 \times r} \end{bmatrix}.$$

For example, if  $\nu = 12$ ,  $\kappa = 4$ , and  $n = 15$ , then

$$A = \left. \begin{array}{l} d \left\{ \begin{array}{l} \begin{array}{c} \overbrace{1 \ 1 \ 1}^d \quad \overbrace{0 \ 0 \ 0 \ 0}^\kappa \quad \overbrace{1 \ 1 \ 1}^{\kappa-1} \quad \overbrace{1 \ 1 \ 1}^{\kappa-1} \quad \overbrace{1 \ 1 \ 1}^r \\ 0 \ 1 \ 1 \quad 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \\ 0 \ 0 \ 1 \quad 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \end{array} \end{array} \right. \\ \kappa-1 \left\{ \begin{array}{l} \begin{array}{c} 1 \ 1 \ 1 \quad 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \\ 1 \ 1 \ 1 \quad 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \\ 1 \ 1 \ 1 \quad 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \end{array} \end{array} \right. \\ \kappa-1 \left\{ \begin{array}{l} \begin{array}{c} 0 \ 1 \ 1 \quad 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \\ 0 \ 1 \ 1 \quad 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \\ 0 \ 1 \ 1 \quad 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \end{array} \end{array} \right. \\ r \left\{ \begin{array}{l} \begin{array}{c} 0 \ 0 \ 1 \quad 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \\ 0 \ 0 \ 1 \quad 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \\ 0 \ 0 \ 0 \quad 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \end{array} \end{array} \right. \\ \kappa \left\{ \begin{array}{l} \begin{array}{c} 0 \ 0 \ 0 \quad 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \\ 0 \ 0 \ 0 \quad 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \\ 0 \ 0 \ 0 \quad 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \end{array} \end{array} \right. \end{array} \right\} \nu$$

Since  $(I_d + U_d)^2 = I_d + U_d$ ,  $BC^t \leq U_d$ ,  $U_d B \leq B$ ,  $BU_\nu \leq B$ ,  $C^t U_d \leq C^t$ ,  $U_\nu C^t \leq C^t$ , and  $C^t B \leq U_\nu^2$ , it follows that

$$A^k = \begin{bmatrix} I_d + U_d & B_{d \times \nu} \\ C_{d \times \nu}^t & C^t B + U_\nu^k \end{bmatrix} \leq A,$$

$$L = \begin{bmatrix} I_d + U_d & B_{d \times \nu} \\ C_{d \times \nu}^t & C^t B \end{bmatrix}, \quad \text{and so} \quad N = \begin{bmatrix} O_{d \times d} & O_{d \times \nu} \\ O_{\nu \times d} & U_\nu \setminus C^t B \end{bmatrix}.$$

(In the  $15 \times 15$  matrix above, the  $I$ 's of  $L$  are the italic  $I$ 's; the remaining  $I$ 's are those of  $N = A \setminus L$ .) The matrix  $U_\nu \setminus C^t B$  is strictly upper-triangular and dominates  $S_\nu$ . Thus  $\nu(A) = \nu$ . Also, all entries of  $C^t B$  on or above the  $\kappa$ th superdiagonal are 1, whereas some entries of the  $(\kappa - 1)$ st superdiagonal are 0. Consequently,  $\kappa$  is the first  $k$  such that  $N^k \leq L$ . Thus,  $\kappa(A) = \kappa$ .

EXAMPLE 2.2. If we allow the additional restriction  $n \geq 2\nu$  (so that the lower bound in Theorem 2.2 equals 1), we can easily describe  $n \times n$  transitive matrices  $A$  with  $\kappa(A) = \kappa$  for each  $1 \leq \kappa \leq \nu$ . Take  $A = I_{n-2\nu} \oplus M$ , where

$$M = \begin{bmatrix} I_\nu + U_\nu & U_\nu^\kappa \\ I_\nu + U_\nu & U_\nu \end{bmatrix},$$

so that

$$M^k = \begin{bmatrix} I_\nu + U_\nu & U_\nu^\kappa \\ I_\nu + U_\nu & U_\nu^k \end{bmatrix} \quad \text{for } 1 \leq k \leq \kappa.$$

Then  $\kappa$  is the first  $k$  such that  $M^k = M^{k+1}$ . Thus,  $\kappa(A) = \kappa$ .

### 3. CHARACTERIZATIONS OF IDEMPOTENCE

Previous results characterizing idempotent binary Boolean matrices can be found in Rosenblatt [8], Schein [9], and Chaudhuri and Mukherjea [3, p. 279]. The last lists six conditions which together characterize idempotence: conditions (ii) and (vi) amount to transitivity. Transitivity is one of the



conditions in Schein's characterization. Rosenblatt's characterization is graph-theoretic; it includes a condition that is very nearly the same as transitivity. Our characterization theorem for idempotent matrices  $A$  includes a limit dominating condition on  $A$  and relates the three matrix parameters  $c(A)$ ,  $b(A)$ ,  $\iota(A)$  defined below to the number  $\tau(A)$  of diagonal  $J$  blocks in the Frobenius normal form of  $A$ . Because the blocks of the Frobenius normal form of  $A$  are either  $J$  blocks or  $O$  blocks (Theorem 1.2),  $\tau(A)$  can be quickly determined from  $A$  by counting the number of distinct columns of  $A$  that have a 1 in the diagonal entry. In graphical terms,  $\tau(A)$  is the number of arc-nonempty strongly connected components of the associated digraph. (A singleton component is counted only if its vertex has a loop.)

The *column rank* of a binary matrix  $M$  is the number  $c(M)$  of vectors in the basis of the space spanned by the columns of  $M$ . (It is an amusing fact that the space spanned by a set of binary vectors has a unique basis, necessarily taken from the original spanning set [4, 6].) The *Boolean rank* of  $M$  is the minimum number  $b(M)$  of rank-1 matrices that sum to  $M$ . (The rank-1 matrices are the same for all notions of rank that we consider in this paper: they are the nonzero outer products  $xy^t$  where  $x, y$  are column vectors.) A set of positions  $(i, j)$  in  $M$  is called *isolated* if each corresponding entry is 1 and no two are in the same row or column and no two are in a  $2 \times 2$  submatrix of  $M$  (equivalently, if no two are in the same row or column and no two are in a  $2 \times 2$  submatrix of  $M$ ). Such a set of positions is more briefly referred to as a set of *isolated* 1's [2]. It is well known that  $c(M) \geq b(M)$  [4, p. 38], and it follows from the definitions above that  $b(M) \geq \iota(M)$ , the maximum number of isolated 1's in  $M$ . Thus, for any binary matrix  $M$ ,

$$c(M) \geq b(M) \geq \iota(M).$$

Each of  $c(M)$ ,  $b(M)$ , and  $\iota(M)$  is unchanged if the rows and columns of  $M$  are permuted. Consequently, when considering these parameters for a limit dominating matrix  $A$ , we may assume that  $A$  is in Frobenius normal form. In fact, since Theorem 1.2 asserts that each block in the Frobenius normal form is either all 0 or all 1, when we consider  $c(A)$ ,  $b(A)$ ,  $\iota(A)$  it turns out that we may restrict our attention to the *contracted* normal form obtained by replacing each block by a single entry.

**THEOREM 3.1.** *A binary matrix  $A$  is idempotent if and only if  $A$  is limit dominating and  $\tau(A)$  equals any one (or all) of  $c(A)$ ,  $b(A)$ ,  $\iota(A)$ .*

*Proof.* If  $A$  is a limit dominating matrix, then  $\tau(A)$  is equal to the number of diagonal 1's in its contracted normal form. Since these diagonal 1's are isolated, we see that the inequalities  $c(A) \geq b(A) \geq \iota(A) \geq \tau(A)$  hold for any limit dominating matrix  $A$ .

If  $A$  is idempotent, then it is certainly limit dominating and  $A = L$ . Assuming that  $A$  is in contracted normal form, the formula for  $A = L$  in Theorem 2.1 implies that each column of  $A$  is a sum of a selection of the  $\tau(A)$  columns  $A_{\cdot j}$  for which  $a_{jj} = 1$ . Thus  $c(A) \leq \tau(A)$ . Therefore  $c(A) = b(A) = \iota(A) = \tau(A)$  if  $A$  is idempotent.

On the other hand, if  $A$  is limit dominating and  $\tau(A)$  equals any of  $c(A), b(A)$ , or  $\iota(A)$ , then  $\tau(A) = \iota(A)$ . Suppose that the limit dominating matrix  $A$  is not idempotent; in particular  $N \neq O$ . As before, we assume that  $A$  is in contracted normal form. Then, by Theorem 2.1,  $L$  can be written as  $\sum_{i=1}^r A_{\cdot j_i} A_{j_i \cdot}$ , where each  $a_{j_i j_i} = 1$ . Each of the rank-1 summands contributes one of the  $\tau$  isolated diagonal 1's to  $A$ . Take any 1 in  $N = A \setminus L$ , say in position  $(p, q)$ . Then  $a_{pp} = 0 = a_{qq}$ ; otherwise the 1 in the  $(p, q)$  position of  $A$  would also be in  $L$ . Thus no pair of the 1's in positions  $(p, q)$  and  $(j_i, j_i)$ ,  $1 \leq i \leq \tau$ , share a row or column. Nor is any such pair in a  $2 \times 2$  submatrix of  $A$ ; otherwise  $a_{pj_i} = 1 = a_{j_i q}$ , and so the  $(p, q)$  entry of both  $N = A \setminus L$  and  $L \geq A_{\cdot j_i} A_{j_i \cdot}$  would be 1. Thus the set of 1's of  $A$  in positions  $(p, q)$  and  $(j_i, j_i)$ ,  $1 \leq i \leq \tau$ , is isolated, so  $\iota(A) > \tau(a)$ , a contradiction. ■

Theorem 3.1 implies that if  $A$  is an idempotent matrix, then  $b(A) = \iota(A)$ . However, the converse is not true even when the matrix is transitive. Let

$$A = \begin{bmatrix} O_n & I_n \\ O_n & O_n \end{bmatrix}.$$

Then  $b(A) = \iota(A) = n$ . Further,  $A$  is transitive, but not idempotent.

#### 4. LIMIT DOMINATING MATRICES OVER A BOOLEAN ALGEBRA

In this section, we generalize our results to limit dominating matrices with entries taken from a finite Boolean algebra  $\mathcal{B}$ . We consider the elements of  $\mathcal{B}$  as the subsets of the set  $\{1, 2, \dots, m\}$ , and we use  $+$  for the union of two elements, juxtaposition for intersection,  $\geq$  for containment, and  $\setminus$  for set difference. We denote the empty set by 0, the set  $\{1, 2, \dots, m\}$  by 1, and the singleton subset (or atom)  $\{i\}$  by  $\alpha_i$  for each  $1 \leq i \leq m$ . Arithmetic for matrices over  $\mathcal{B}$  is defined as it usually is—entrywise addition and scalar multiplication, and the row-column rule for matrix multiplication. Also, if  $X$  and  $Y$  are matrices over  $\mathcal{B}$  of the same size, we say that  $X$  *dominates*  $Y$  and write  $X \geq Y$  if each entry of  $X$  contains the corresponding entry of  $Y$ ; we

then let  $X \setminus Y$  denote the matrix obtained by taking set differences entry-wise.

Given any matrix  $M$  with entries from  $\mathcal{B}$ , we define the  $i$ th *constituent* of  $M$  to be the  $\{0, 1\}$  matrix  $M_i$  such that  $\alpha_i M = \alpha_i M_i$  for each  $1 \leq i \leq m$ . Note that  $M = \sum_{i=1}^m \alpha_i M_i$ . Moreover, for two matrices  $X$  and  $Y$  over  $\mathcal{B}$  of the same shape,  $X = Y$  ( $X \geq Y$ ) if and only if  $X_i = Y_i$  ( $X_i \geq Y_i$ ) for all  $1 \leq i \leq m$ . Also, if  $X$  and  $Y$  are compatible with the operations considered, then  $(X + Y)_i = X_i + Y_i$ ,  $(X \setminus Y)_i = X_i \setminus Y_i$ ,  $(XY)_i = X_i Y_i$ , and  $(X^k)_i = (X_i)^k$  for all  $1 \leq i \leq m$ . In particular, we note that the notation  $X_i^k$  is well defined. For more details on constituent matrices, see [5].

As before, we say that a square matrix  $A$  over  $B$  is *limit dominating* if its powers converge to a limit dominated by  $A$ . It follows from the properties above that  $A$  is limit dominating if and only if each of its constituents  $A_i$  is. Moreover, the powers  $A^k = \sum_{i=1}^m \alpha_i A_i^k$  converge to the limit  $L = \sum_{i=1}^m \alpha_i L_i$  where for each  $i$ ,  $L_i$  is the limit of the powers of the limit dominating binary constituent  $A_i$ . We can now generalize Theorem 2.1. As before,  $A_j$  and  $A_{\cdot j}$  denote the  $j$ th row and column of  $A$ , respectively.

**THEOREM 4.1.** *Suppose that  $A$  is an  $n \times n$  limit dominating matrix over  $\mathcal{B}$  and that  $L$  is the limit of the powers of  $A$ . Let  $N$  be the residual matrix  $A \setminus L$ . Then*

$$A^k = L + N^k \quad \text{for all } k \geq 1,$$

*and  $N$  is nilpotent. Moreover,  $L = \sum_{j=1}^n a_{jj} A_{\cdot j} A_j$ .*

*Proof.* By Theorem 2.1, each of the constituents  $N_i = A_i \setminus L_i$  of  $N$  is nilpotent and  $A_i^k = L_i + N_i^k$  for all  $k \geq 1$ . Thus  $N$  is nilpotent and  $A^k = L + N^k$  for all  $k \geq 1$ . Let  $a_{jj}^i$  denote the  $j$ th diagonal entry of  $L_i$ . Applying Theorem 2.1 to each constituent  $L_i$  in  $L = \sum_{i=1}^m \alpha_i L_i$ , we get

$$\begin{aligned} L &= \sum_{i=1}^m \alpha_i \sum_{j=1}^n a_{jj}^i (A_i)_{\cdot j} (A_i)_j \\ &= \sum_{j=1}^n \sum_{i=1}^m \alpha_i a_{jj}^i (A_i)_{\cdot j} (A_i)_j = \sum_{j=1}^n a_{jj} A_{\cdot j} A_j. \quad \blacksquare \end{aligned}$$

We define the index of convergence  $\kappa(A)$  of  $A$  and the index of nilpotence  $\nu(A)$  of its residual  $N$  as in Section 2:

$$\kappa(A) = \min\{k : A^k = L\} \quad \text{and} \quad \nu(A) = \min\{k : N^k = O\}.$$

Because  $A^k = \sum_{i=1}^m \alpha_i A_i^k$  and  $N^k = \sum_{i=1}^m \alpha_i N_i^k$ , we have

$$\kappa(A) = \max_{1 \leq i \leq m} \kappa(A_i) \quad \text{and} \quad \nu(A) = \max_{1 \leq i \leq m} \nu(A_i).$$

Our bounds on the index of convergence from Section 2 extend to limit dominating matrices over  $\mathcal{B}$ .

**THEOREM 4.2.** *If  $A$  is an  $n \times n$  limit dominating matrix over  $\mathcal{B}$ , then*

$$\nu(A) \geq \kappa(A) \geq \left\lceil \frac{n}{n - \nu(A) + 1} \right\rceil.$$

*Proof.* The upper bounds on  $\kappa(A)$  follows from Theorem 4.1. We have  $\kappa(A) \geq \kappa(A_i)$  for  $1 \leq i \leq m$ . Thus,  $\kappa(A) \geq \lceil n/(n - \nu(A_i) + 1) \rceil$  for  $1 \leq i \leq m$ , by the binary case. As  $\nu(A) = \nu(A_i)$  for some  $i$ , the lower bound also holds.

We say that a nonzero matrix over  $\mathcal{B}$  has rank 1 if it is of the form  $xy^t$  where  $x$  and  $y$  are column vectors over  $\mathcal{B}$ . As before, we define the Boolean rank of a matrix  $M$  over  $\mathcal{B}$  to be the minimum number  $b(M)$  of rank-1 matrices that sum to  $M$ . It turns out that the Boolean rank of  $M$  is the maximum of the (binary) Boolean ranks of its constituents:

$$b(M) = \max_{1 \leq i \leq m} b(M_i).$$

The column rank of  $M$  is defined as the number  $c(M)$  of vectors in a basis of the space spanned by the columns of  $M$ . Also, (see [5, Corollary 2.2.1]),

$$c(M) = \max_{1 \leq i \leq m} c(M_i).$$

Since a matrix  $A$  over  $\mathcal{B}$  is limit dominating (or idempotent) if and only if each of its constituents is, we have the following immediate extension of Theorem 3.1.

**THEOREM 4.3.** *A matrix  $A$  over  $\mathcal{B}$  is idempotent if and only if  $A_i$  is limit dominating for all  $1 \leq i \leq m$  and  $\tau(A_i)$  equals any one (or all) of  $c(A_i)$ ,  $b(A_i)$ ,  $\nu(A_i)$  for each  $1 \leq i \leq m$ .*

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